Partial Differential Equations:

Basic Concepts:

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called a partial differential equation. The order of the highest derivative is called the order of the equation.

Just as in the case of an ordinary differential equation, we say that a partial differential equation is linear if it is of the first degree in the dependent variable (the unknown function) and its partial derivatives. If each term of such term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous; otherwise it is said to be nonhomogeneous.

Example:

Important linear partial differential equation of the second order:

$$(1)\ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional wave Eq.}$$

$$(2)\ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional heat Eq.}$$

$$(3)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textit{Two-dimensional Laplace Eq.}$$

$$(4)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \textit{Two-dimensional Poisson Eq.}$$

$$(5)\ \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \textit{Two-dimensional wave Eq.}$$

$$(6)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \textit{Three-dimensional Laplace Eq.}$$

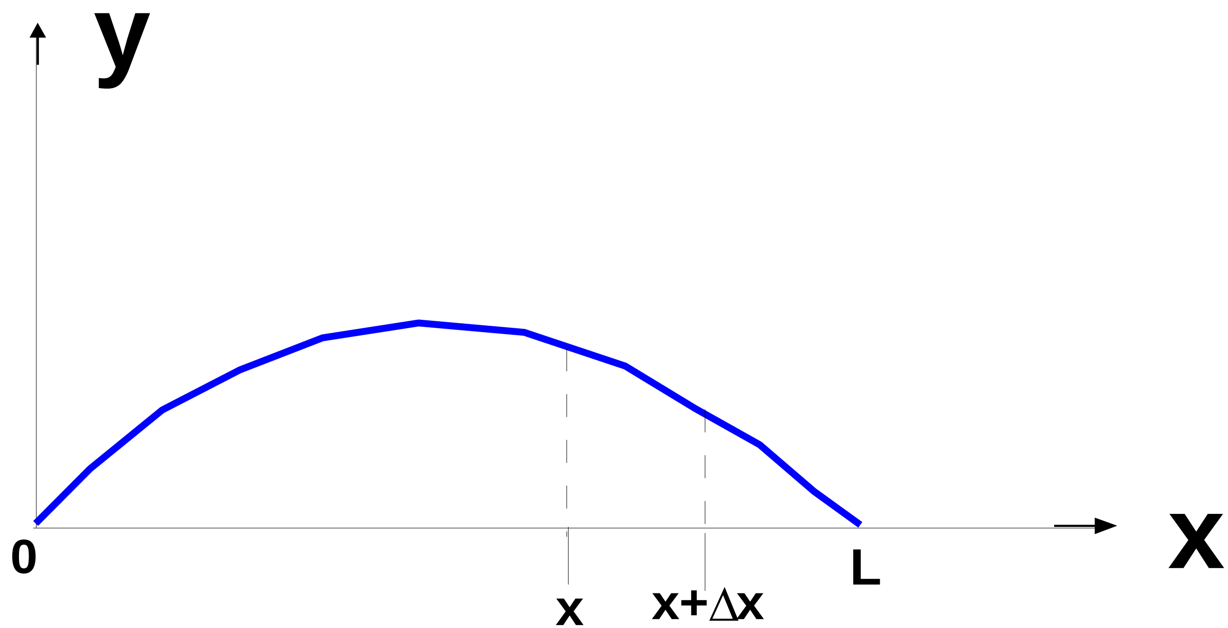
Here (c) is a constant; (t) is a time; x, y, z are the Cartesian coordinates and the dimension is the number of these coordinates in the equation. Equation (4) [with $f(x,y) \neq 0$] is nonhomogeneous, while the other equations are homogeneous.

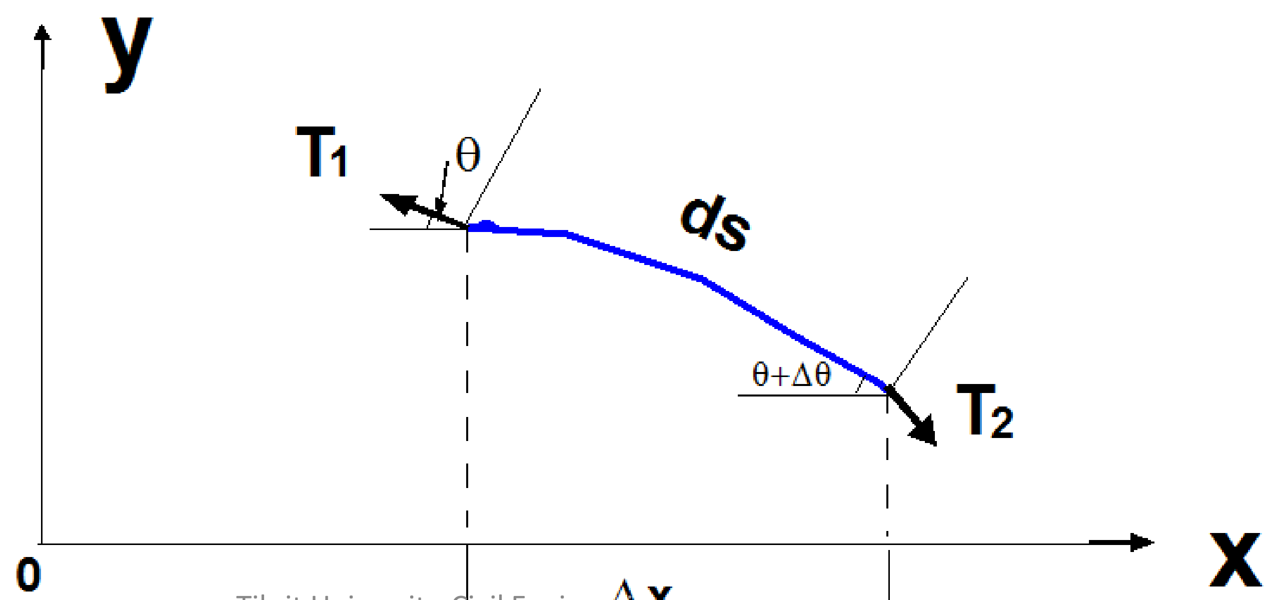
MODELING:

(1)Vibration of elastic string (wave equation):

Assumptions;

- Vibration takes place in x-y plane.
- No elongation.
- The string can transmit force only in the direction of its length.
- Constant tension force.


y
0
x
x+Δx
L
X


y
T1
θ
ds
θ+Δθ
T2
0
Δx
X

$$\sum Fx = 0$$

$$T_2 \cos(\theta + \Delta\theta) - T_1 \cos\theta = 0$$

$$\theta \ is \ \ too \ \ small \quad \theta \to 0$$

$$T_2 - T_1 = 0$$

$$\therefore T_2 = T_1 = T$$

$$\sum Fy = 0$$

$$T_1 \sin \theta - T_2 \sin(\theta + \Delta\theta) - \rho \Delta s(-\frac{\partial^2 y}{\partial t^2}) = 0$$

$$\rho = density \ per \ length$$

$$T_1 \sin\theta - T_2 \sin(\theta + \Delta\theta) - \rho\,\Delta s\left(-\frac{\partial^2 y}{\partial t^2}\right) = 0$$

$$\left.\begin{aligned} \sin\theta &= \theta \\ \sin(\theta + \Delta\theta) &= \theta + \Delta\theta \end{aligned}\right\} \quad \theta \;\; is \;\; too \;\; small$$

$$T_1 = T_2 = T$$

$$T\ \theta - T\ (\theta + \Delta\theta) = -\rho\,\Delta s\,\frac{\partial^2 y}{\partial t^2}$$

$$\therefore T\Delta\theta = \rho\,\Delta s\,\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\Delta\theta}{\Delta s} = \frac{\rho}{T}\,\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\Delta\theta}{\Delta s} = curvature$$

$$= \frac{\partial^2 y / \partial x^2}{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}} \approx \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\rho}{T} = a^2$$

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2} \qquad One-\dim ensional\; wave\; Eq.$$

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2} \qquad (1)$$

By separation the variables:

$$y = X\ T$$

$$X = function\ \ of\ \ x$$

$$T = function\ \ of\ \ t$$

$$\therefore \left.\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= X''T \\ \frac{\partial^2 y}{\partial t^2} &= X\ T'' \end{aligned}\right\} Subst.\ into\ \ Eq.\ (1)$$

$$X''\ T = a^2\ X\ \ T''$$

$$X''\,T = a^2\,X\;T''$$

$$\frac{X''}{X} = a^2\,\frac{T''}{T}$$

$$\frac{X''}{a^2 X} = \frac{T''}{T} = Constant \rightarrow \begin{cases} -k^2 \\ 0 \\ k^2 \end{cases}$$

$$(1)\quad \frac{X''}{a^2 X} = \frac{T''}{T} = -k^2$$

$$X'' + a^2 k^2 X = 0 \Leftrightarrow \frac{d^2 X}{dx^2} + a^2 k^2 X = 0$$

$$X = c_1 \cos kax + c_2 \sin kax$$

$$\frac{T''}{T} = -k^2$$

$$T'' + k^2 T = 0 \;\Leftrightarrow\; \frac{d^2T}{dt^2} + k^2 T = 0$$

$$T = c_3 \cos k\,t + c_4 \sin k\,t$$

$$\therefore\; y(x,t) = X.T$$

$$= (c_1 \cos k\,a\,x + c_2 \sin k\,a\,x)(c_3 \cos k\,t + c_4 \sin k\,t)$$

(2) $Constant = 0$

$$\frac{X''}{a^2 X} = \frac{T''}{T} = 0$$

$$\frac{X''}{a^2 X} = 0 \rightarrow X'' = 0 \rightarrow X' = c_1 \rightarrow X = c_1 x + c_2$$

$$\frac{T''}{T} = 0 \rightarrow T'' = 0 \rightarrow T' = c_3 \rightarrow T = c_3 t + c_4$$

$$\therefore y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$$

$$(3)\ Constant = k^2$$

$$\frac{X''}{a^2X} = \frac{T''}{T} = k^2$$

$$X'' + a^2k^2X = 0 \Leftrightarrow \frac{d^2X}{dx^2} - a^2k^2X = 0$$

$$X = c_1e^{kax} + c_2e^{-kax}$$

$$\frac{T''}{T} = k^2$$

$$T'' + k^2T = 0 \Leftrightarrow \frac{d^2T}{dx^2} - k^2T = 0$$

$$T = c_3e^{kt} + c_4e^{-kt}$$

$$\therefore y(x,t) = (c_1e^{kax} + c_2e^{-kax})(c_3e^{kt} + c_4e^{-kt})$$

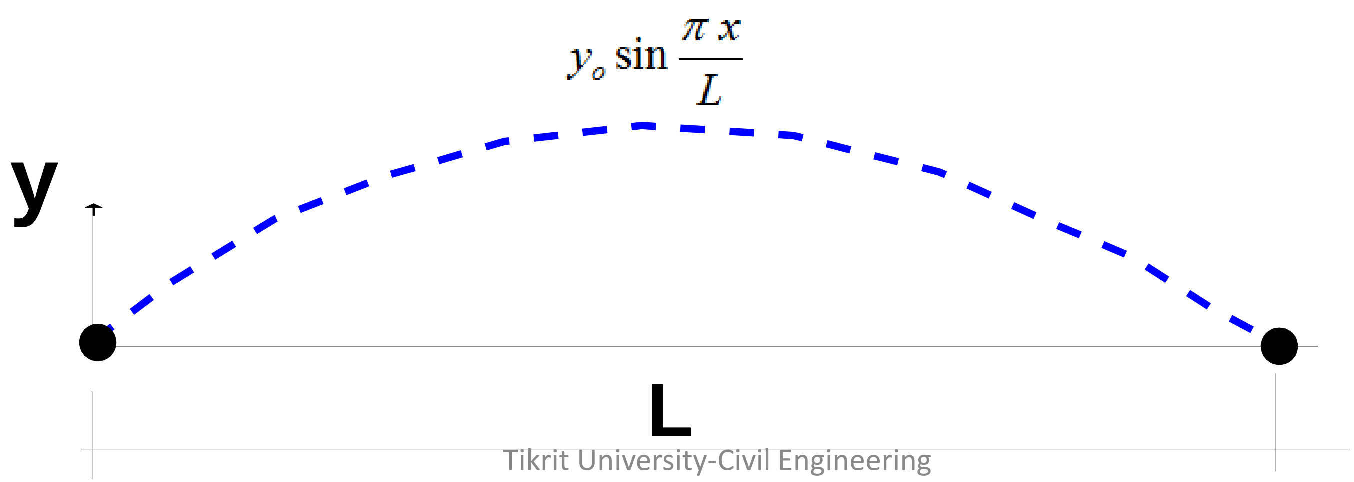
For the string fixed at the both ends and subjected to the following:

<u>*Initial condition:*</u>

$$y(x,0) = y_o \sin\frac{\pi x}{L}$$

$$\left.\frac{\partial y}{\partial t}\right)(x,0) = 0$$

$$y(x,t) = (c_1 \cos k\,a\,x + c_2 \sin k\,a\,x)(c_3 \cos k\,t + c_4 \sin k\,t)$$



$$y(x,t) = (c_1 \cos k a x + c_2 \sin k a x)(c_3 \cos k t + c_4 \sin k t)$$

$$y(0,t) = 0 \quad \& \quad y(L,t) = 0$$

$$0 = (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos k t + c_4 \sin k t)$$

$$c_1(c_3 \cos k t + c_4 \sin k t) = 0$$

$$Either; \quad c_1 = 0$$

$$Or; \quad (c_3 \cos k t + c_4 \sin k t) = 0$$

$$\therefore\ y(x,t) = c_2 \sin k a x(c_3 \cos k t + c_4 \sin k t)$$

$$Or; y(x,t) = \sin k a x(A \cos k t + B \sin k t)$$

$$Such\ \ that; \ c_2 * c_3 = A \quad \& \quad c_2 * c_4 = B$$

$$y(x,t)=\sin k\,a\,x(A\cos k\,t+B\sin k\,t)$$

$$y(L,t)=0$$

$$0=\sin k\,a\,L(A\cos k\,t+B\sin k\,t)$$

$$\sin k\,a\,L=0 \;\Rightarrow\; k\,a\,L=n\pi \;\Rightarrow\; \therefore k=\frac{n\pi}{a\,L}$$

$$y(x,t)=\sum_{n=1}^{\infty}\sin k\,a\,x(A_n\cos k\,t+B_n\sin k\,t)$$

$$y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi}{a\,L}t+B_n\sin\frac{n\pi}{a\,L}t\right)$$

Initialconditions:

$$t = 0 \Rightarrow y(x,t) = y_o \sin\frac{\pi x}{L}$$

$$y_o \sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(A_n \cos 0 + B_n \sin 0)$$

$$y_o \sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$A_n = \frac{2}{L}\int_0^L y_o \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}dx$$

$$A_n = \frac{2}{L}y_o L = 2y_o$$

$$\frac{\partial y}{\partial t}(x,0)=0$$

$$y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi}{a\,L}t+B_n\sin\frac{n\pi}{a\,L}t\right)$$

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(-\frac{n\pi}{a\,L}A_n\sin\frac{n\pi}{a\,L}t+\frac{n\pi}{a\,L}B_n\cos\frac{n\pi}{a\,L}t\right)$$

$$At\ \ t=0\ ;\ \ \frac{\partial y}{\partial t}=0$$

$$0=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(-\frac{n\pi}{a\,L}A_n\sin 0+\frac{n\pi}{a\,L}B_n\cos 0\right)$$

$$0=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(\frac{n\pi}{a\,L}B_n\right)\ \Rightarrow\ \underline{B_n=0}$$

$$\therefore\ y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\,2y_o\cos\frac{n\pi}{a\,L}t$$

$(2) One-\dim e nsional consolidation$:

$$\frac{\partial^2 u}{\partial z^2} = a^2 \frac{\partial \quad u}{\partial t}$$

$$u(z,t) = Z_z T_t$$

$$\frac{\partial^2 u}{\partial z^2} = Z''T$$

$$\frac{\partial u}{\partial t} = ZT'$$

$$Z''T = a^2 ZT'$$

$$\frac{Z''}{a^2 Z} = \frac{T'}{T} = Cons\tan t = \begin{cases} -k^2 \\ 0 \\ k^2 \end{cases}$$

$$\frac{Z''}{a^2 Z} = -k^2$$

$$Z'' + a^2 k^2 Z = 0 \Leftrightarrow \frac{d^2 Z}{dx^2} + a^2 k^2 Z = 0$$

$$Z = c_1 \cos k\,a\,z + c_2 \sin k\,a\,z$$

$$\frac{T'}{T} = -k^2$$

$$T' + k^2 T = 0 \Leftrightarrow \frac{dT}{dt} + k^2 T = 0$$

$$T = c_3 e^{-k^2 t}$$

$$\therefore u(z,t) = Z.T$$

$$= (c_1 \cos k\,a\,z + c_2 \sin k\,a\,z)\ c_3\, e^{-k^2 t}$$

$$u(z,t) = c_3 e^{-k^2 t}(c_1 \cos k a z + c_2 \sin k a z)$$

$$Let \quad A = c_1 * c_2 \quad \& \quad B = c_1 * c_3$$

$$\therefore \underline{\underline{u(z,t) = e^{-k^2 t}(A \cos k a z + B \sin k a z)}} \quad general\ Sol.$$

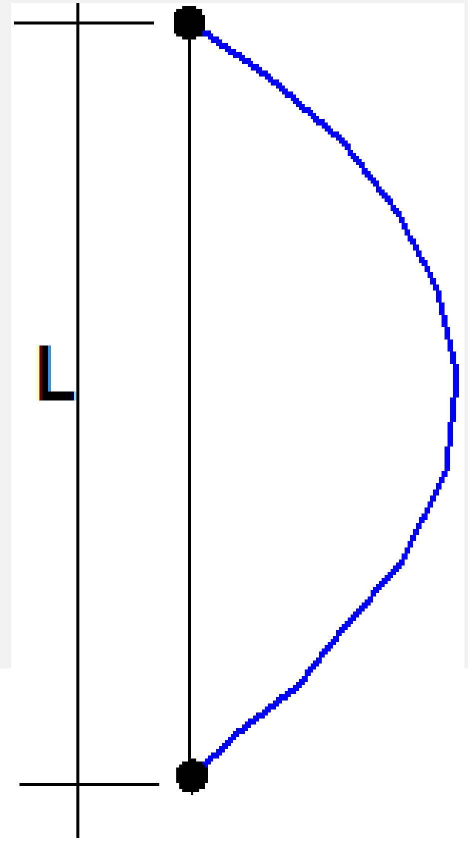
$\underline{Boundary\ Conditions:}$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$\underline{Initial\ Condition:}$

$$u(z,0) = f(z)\ given$$



$\underline{A\ clay\ \ layer:}$

$$u(z,0) = z(L - z)$$

$$u(z,t) = e^{-k^2 t}(A\cos k\,a\,z + B\sin k\,a\,z)$$

$$(1)u(o,t) = 0$$

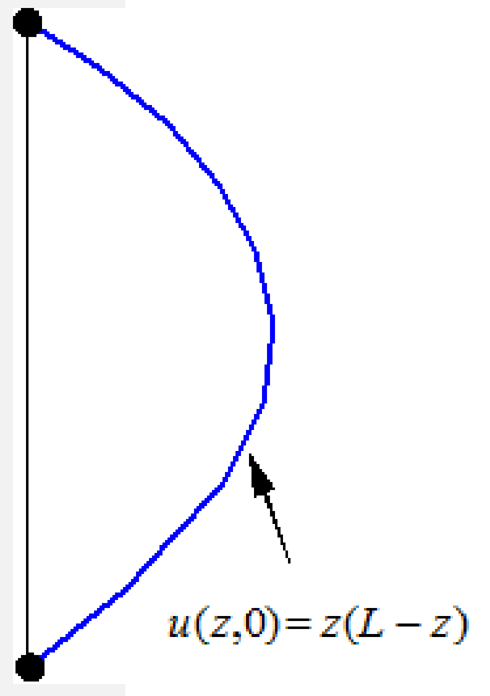
$$0 = e^{-k^2 t}(A\cos 0 + B\sin 0)$$

$$0 = e^{-k^2 t} A \;\;\Rightarrow\;\; A = 0$$

$$(1)u(L,t) = 0$$

$$0 = e^{-k^2 t}\;\; B\sin k\,a\,L$$

$$B \neq 0$$



$$\therefore \sin k\,a\,L = 0 \;\;\Rightarrow\;\;\; k\,a\,L = n\pi \Rightarrow k = \frac{n\pi}{a\,L}$$

$$u(z,t)=\sum_{n=1}^{\infty} B \sin k a z\, e^{-k^2 t}$$

$$=\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} z \;\; e^{-k^2 t}$$

$$u(z,0)=z(L-z)$$

$$z(L-z)=\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} z \;\; e^{0}$$

$$z(L-z)=\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} z$$

$$Bn=\frac{2}{L}\int_0^L z(L-z)\sin \frac{n\pi z}{L}\, dz$$

$$\vdots$$

$$\vdots$$

H. W.

Solution of Partial Differential Equations:

Here under we consider the simple examples, the solution of which depends up to the meaning of partial differentiation.

Example (1):

Solve; (i) $\frac{\partial z}{\partial x}=0$

(ii) $\frac{\partial^2 z}{\partial y^2}=0$

(iii) $\frac{\partial^3 z}{\partial x^2 \partial y}=0$

Noting that $\underline{z}$ is a function of $\underline{x \ \& \ y}$.

Solution:

$$(i)\ \frac{\partial z}{\partial x} = 0 \qquad (a)$$

Here z is a function of two independent variables x and y. On integrating Eq. (a), we get that solution as;

$z =$ function independent of x.

$\therefore z = \phi(y)$ which is arbitrary.

$$(ii)\ \frac{\partial^2 z}{\partial y^2} = 0$$

z *is a function* x *and* y.

Integrating with respect to y, *we get*;

$$\frac{\partial z}{\partial y} = a \text{ function independent of } y$$

$$= \phi(x) \quad (\ldots \text{the other variable is } x \text{ only})$$

Again integrating with respect to y, *we get*;

$$z = \int \phi(x)\, dy + a \text{ function independent of } y$$

$$= \phi(x) \int dy + f(x)$$

$$\therefore z = \phi(x)\, y + f(x) \text{ where } \phi(x) \text{ and } f(x) \text{ are arbitrary in the solution}$$

$$(iii)\ \frac{\partial^3 z}{\partial x^2 \partial y} = 0$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\{\frac{\partial^2 z}{\partial x^2}\} = 0$$

On integrating with respect to y, we get;

$$\frac{\partial^2 z}{\partial x^2} = a\ function\ independent\ of\ \ y$$

$$= \phi(x) \qquad\qquad\qquad (say)$$

$$Now;\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\{\frac{\partial z}{\partial x}\} = \phi(x)$$

$$Now;\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\{\frac{\partial z}{\partial x}\} = \phi(x)$$

$On\ \text{int}egrating\ with\ respect\ to\ x,\ we\ get;$

$$\frac{\partial z}{\partial x} = \int \phi(x)\,dx + a\ function\ independent\ of\ \ x$$

$$= \int \phi(x)dx + q(y)$$

$Again\ \text{int}egrating\ with\ respect\ to\ x,\ we\ get;$

$$\therefore z = [\int \phi(x)dx]dx + \int q(y)\,dx + F(y)$$

$\underline{\underline{Or:}}$

$$z = \int\int \phi(x)dxdx + \ x\,q(y) + F(y)\ which\ is\ required\ solution$$

Example (2): Solve the partial differential equations in the following case;

$$(i)\ \frac{\partial^2 z}{\partial x^2} = \sin x$$

$$(ii)\ \frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x \quad \text{given that } u = 0 \text{ when } t = 0$$

$$\text{and } \frac{\partial u}{\partial t} = 0 \text{ when } x = 0$$

$$(iii)\ \frac{\partial^2 z}{\partial x \partial y} = \sin x \ \sin y \quad \text{for which } \frac{\partial z}{\partial y} = -2\sin y \text{ when } x = 0$$

$$\text{and } z = 0 \text{ when } y \text{ is an odd multiply of } \frac{\pi}{2}$$

$$(iv)\ \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x+y} \quad \text{given that } z = y\ln y \text{ and}$$

$$\frac{\partial z}{\partial y} = 1 + \ln y \quad \text{when } x = 0$$

Solution:

$$(i)\ \frac{\partial^2 z}{\partial x^2} = \sin x$$

Integrating w.r.t. x, we get;

$$\frac{\partial z}{\partial x} = \int \sin x\, dx + f(y) = -\cos x + f(y)$$

Again integrating w.r.t. x, we get;

$$z = -\int \cos x\, dx + \int f(y)\, dx + g(y)$$

$$= -\sin x + x f(y) + g(y)$$

where $f(y)$ and $g(y)$ are arbitrary functions.

$$(ii)\ \frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \qquad \begin{cases} u = 0 & when \ \ t = 0 \\ \dfrac{\partial u}{\partial t} = 0 & when \ \ x = 0 \end{cases}$$

Integrating partially w.r.t. x, we get;

$$\frac{\partial u}{\partial t} = \int e^{-t} \cos x \, dx + a \ function \ independent \ of \ (x)$$

$$= e^{-t} \int \cos x \, dx + f(t) \qquad (say)$$

$$\therefore \frac{\partial u}{\partial t} = e^{-t} \sin x + f(t)$$

$$x = 0 \Rightarrow \frac{\partial u}{\partial t} = 0$$

$$0 = e^{-t} \sin 0 + f(t) \Rightarrow f(t) = 0$$

$$\therefore \frac{\partial u}{\partial t} = e^{-t} \sin x$$

$$\frac{\partial u}{\partial t} = e^{-t} \sin x$$

Now integrating partially w.r.t. t, we get;

$$u(x,t) = \int e^{-t} \sin x \, dt + a \text{ function independent of } (t)$$

$$= \sin x \int e^{-t} \, dt + g(x) \qquad (say)$$

$$u(x,t) = -e^{-t} \sin x + g(x)$$

$$t = 0 \Rightarrow u = 0$$

$$0 = -e^{0} \sin x + g(x) \Rightarrow f(x) = \sin x$$

$$\therefore u(x,t) = -e^{-t} \sin x + \sin x = \underline{\underline{\sin x (1 - e^{-t})}}$$

$$(iii)\ \frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \qquad \textit{for which } \frac{\partial z}{\partial y} = -2\sin y \textit{ when } x = 0$$

$$\textit{and } z = 0 \textit{ when } y \textit{ is an odd multiply of } \frac{\pi}{2}$$

Integrating partially w.r.t. x, we get;

$$\frac{\partial z}{\partial y} = \sin y \int \sin x\, dx + f(y)$$

$$= \sin y(-\cos x) + f(y)$$

$$x = 0 \Rightarrow \frac{\partial z}{\partial y} = -2\sin y$$

$$-2\sin y = \sin y(-\cos 0) + f(y)$$

$$-2\sin y = \sin y(-1) + f(y) \Rightarrow f(y) = -\sin y$$

$$\therefore \frac{\partial z}{\partial y} = -\sin y \cos x - \sin y = -\sin y(\cos x + 1)$$

$$\frac{\partial z}{\partial y} = -(\cos x + 1)\sin y$$

Now integrating partially w.r.t. y, we get;

$$z = -(\cos x + 1)\int \sin y \, dy + g(x)$$

$$= (\cos x + 1)\cos y + g(x)$$

$$z = 0 \Rightarrow \quad y = odd * \frac{\pi}{2}$$

$$0 = (\cos x + 1)(0) + g(x) \Rightarrow \quad g(x) = 0$$

$$\therefore z = (\cos x + 1)\cos y$$

$$(iv)\ \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x+y} \qquad given\ that\ z = y\ln y\ and$$

$$\frac{\partial z}{\partial y} = 1 + \ln y \quad when\ \ x = 0$$

Integrating partially w.r.t. x, we get;

$$\frac{\partial z}{\partial y} = \int \frac{1}{x+y}\,dx + f(y)$$

$$= \ln(x+y) + f(y)$$

$$x = 0 \Rightarrow \frac{\partial z}{\partial y} = 1 + \ln y$$

$$1 + \ln y = \ln(0+y) + f(y)$$

$$\therefore\ f(y) = 1 + \ln y - \ln y = 1$$

$$\therefore \frac{\partial z}{\partial y} = \ln(x+y) + 1$$

$$\frac{\partial z}{\partial y} = \ln(x+y)+1$$

Now integrating partially w.r.t. y, we get;

$$z = \int \ln(x+y)\,dy + \int 1\,dy + g(x)$$

$$\int u\,dv = uv - \int v\,du \qquad \{\ln(x+y) = u \quad and \quad dy = dv)$$

$$z = \ln(x+y)\,y - \int (\frac{1}{x+y}\,y)\,dy + y + g(x)$$

$$= y\ln(x+y) - \int (1 - \frac{x}{x+y})\,dy + y + g(x)$$

$$= y\ln(x+y) - \int 1\,dy + x\int \frac{dy}{x+y} + y + g(x)$$

$$z = y\ln(x+y) - \int 1dy + x\int \frac{dy}{x+y} + y + g(x)$$

$$= y\ln(x+y) - y + x\ln(x+y) + y + g(x)$$

$$= y\ln(x+y) + x\ln(x+y) + g(x)$$

$$\therefore z(x,y) = (x+y)\ln(x+y) + g(x)$$

$$x = 0 \quad \Rightarrow z = y\ln y$$

$$y\ln y = (0+y)\ln(0+y) + g(x) \Rightarrow g(x) = 0$$

$$\therefore z = (x+y)\ln(x+y)$$